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A class of mixed integrable models

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Abstract
The algebraic structure of the integrable mixed mKdV/sinh-Gordon model is discussed and extended to the AKNS/Lund–Regge model and to its corresponding supersymmetric versions. The integrability of the models is guaranteed from the zero curvature representation and some soliton solutions are discussed.

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1. Introduction

The mKdV and sine-Gordon equations are nonlinear differential equations belonging to the same integrable hierarchy representing different time evolutions [1]. The structure of its soliton solutions present the same functional form in terms of

$$\rho = \mathrm{e}^{kx+k^n t_n}, \tag{1.1}$$

which carries the spacetime dependence. Solutions of different equations within the same hierarchy differ only by the factor $k^n t_n$ in ρ. For instance $n = 3$ corresponds to the mKdV equation and $n = -1$ to the sinh-Gordon. For $n > 0$ a systematic construction of integrable hierarchies can be solved and classified according to a decomposition of an affine Lie algebra, $\hat{\mathcal{G}}$ and a choice of a semi-simple constant element E (see [2] for review). Such a framework was shown to be derived from the Riemann–Hilbert decomposition which was later shown to incorporate negative grade isospectral flows $n < 0$ [3] as well.

The mixed system

$$\phi_{xt} = \frac{\alpha_3}{4}\left(\phi_{xxxx} - 6\phi_x^2\phi_{xx}\right) + 2\eta\sinh(2\phi) \tag{1.2}$$

is a nonlinear differential equation which represents the well-known mKdV equation for $\eta = 0$ ($v = -\partial_x\phi$) and the sinh-Gordon equation for $\alpha_3 = 0$. It was introduced in [4] where, employing the inverse scattering method, multi-soliton solutions were constructed by modification of time dependence in ρ. Solutions (multi-soliton) were also considered in [5] by Hirota's method. Moreover, a two-breather solution was discussed in [6] in connection with few-optical-cycle pulses in transparent media. The soliton solutions obtained in [4–6] indicates integrability of the mixed model (1.2).

In this paper, we consider the mixed system mKdV/sinh-Gordon (1.2) within the zero curvature representation. We show that a systematic solution for the mixed model is obtained by the dressing method and a specific choice of vacuum solution. Such formalism is extended to the mixed AKNS/Lund–Regge and to its supersymmetric versions as well.

In the last section, we discuss the coupling of higher positive and negative flows generalizing the examples given previously.

2. The mixed mKdV/sinh-Gordon model

Let us consider a nonlinear system composed of a mixed sinh-Gordon and mKdV equation given by equation (1.2) and the following zero curvature representation,

$$\left[\partial_x + E^{(1)} + A_0, \partial_t + D_3^{(3)} + D_3^{(2)} + D_3^{(1)} + D_3^{(0)} + D_3^{(-1)}\right] = 0 \tag{2.1}$$

where $E^{(2n+1)} = \lambda^n(E_\alpha + \lambda E_{-\alpha})$, $A_0 = vh$ and $E_{\pm\alpha}$ and h are $sl(2)$ generators satisfying $[h, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}$, $[E_\alpha, E_{-\alpha}] = h$. According to the grading operator $Q = 2\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda} + \frac{1}{2}h$, $D_3^{(j)}$ is a graded j Lie algebra valued and equation (2.1) decomposes into six independent equations (decomposing grade by grade):

$$\begin{aligned}
&\left[E, D_3^{(3)}\right] = 0,\\
&\left[E, D_3^{(2)}\right] + \left[A_0, D_3^{(3)}\right] + \partial_x D_3^{(3)} = 0,\\
&\left[E, D_3^{(1)}\right] + \left[A_0, D_3^{(2)}\right] + \partial_x D_3^{(2)} = 0,\\
&\left[E, D_3^{(0)}\right] + \left[A_0, D_3^{(1)}\right] + \partial_x D_3^{(1)} = 0,\\
&\left[E, D_3^{(-1)}\right] + \left[A_0, D_3^{(0)}\right] + \partial_x D_3^{(0)} - \partial_t A_0 = 0,\\
&\left[A_0, D_3^{(-1)}\right] + \partial_x D_3^{(-1)} = 0.
\end{aligned} \tag{2.2}$$

where $E \equiv E^{(1)}$. In order to solve (2.2) let us propose

$$\begin{aligned}
D_3^{(3)} &= \alpha_3(\lambda E_\alpha + \lambda^2 E_{-\alpha}) + \beta_3(\lambda E_\alpha - \lambda^2 E_{-\alpha}),\\
D_3^{(2)} &= \sigma_2 \lambda h,\\
D_3^{(1)} &= \alpha_1(E_\alpha + \lambda E_{-\alpha}) + \beta_1(E_\alpha - \lambda E_{-\alpha}),\\
D_3^{(0)} &= \sigma_0 h.
\end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2) we obtain $\beta_3 = 0$, $\alpha_3 = \text{const}$ and

$$\beta_1 = \frac{\alpha_3}{2} v_x, \qquad \alpha_1 = -\frac{\alpha_3}{2} v^2, \qquad \sigma_0 = \frac{\alpha_3}{4}(v_{xx} - 2v^3), \qquad \sigma_2 = \alpha_3 v. \tag{2.4}$$

In order to solve the last equation in (2.2) we parametrize

$$A_0 = -\partial_x B B^{-1} = -\partial_x \phi h, \qquad B = \mathrm{e}^{\phi h} \tag{2.5}$$

and

$$D_3^{(-1)} = \eta B E^{(-1)} B^{-1} = \eta \lambda^{-1}(\mathrm{e}^{2\phi} E_\alpha + \lambda\, \mathrm{e}^{-2\phi} E_{-\alpha}). \tag{2.6}$$

The zero grade projection in (2.2) yields the time evolution equation (1.2). Note that in order to solve the last equation (2.3) we have introduced the sinh-Gordon variable ϕ in (2.5) and (2.6) such that $v = -\partial_x \phi$.

Let us now recall some basic aspects of the dressing method which provides systematic construction of soliton solutions. The zero curvature representation implies in a pure gauge configuration. In particular, the vacuum is obtained by setting $\phi_{\text{vac}} = 0$ or $\upsilon_{\text{vac}} = 0$ which, when in (2.1) implies

$$\partial_x T_0 T_0^{-1} = E^{(1)}, \qquad \partial_t T_0 T_0^{-1} = \alpha_3 E^{(3)} + \eta E^{(-1)} \tag{2.7}$$

and after integration

$$T_0 = \exp(t(\alpha_3 E^{(3)} + \eta E^{(-1)}))\exp(x E^{(1)}), \qquad E^{(2n+1)} = \lambda^n (E_\alpha + \lambda E_{-\alpha}). \tag{2.8}$$

If we identify $\upsilon = -\partial_x \phi$ equation (1.2) represents a coupling of mKdV and sinh-Gordon equations and becomes a pure mKdV when $\eta = 0$ and pure sinh-Gordon when $\alpha_3 = 0$. Tracing back those two limits from (2.4) and (2.6) it becomes clear that the sinh-Gordon limit ($\eta = 0$) in (1.2) is responsible for the vanishing of $D_3^{(-1)}$. On the other hand, $\alpha_3 = 0$ implies $D_3^{(j)} = 0, j = 0, \ldots, 3$. Inspired by the dressing method for constructing soliton solutions of integrable hierarchies (see for instance [7]) and the fact that the nth member of the hierarchy is associated with the time evolution parameter $k_i^n t_n$ ($n = 3$ for mKdV and $n = -1$ for sinh-Gordon) it is natural to propose soliton solutions based on the modified spacetime dependence

$$\rho_i = \exp\left(2k_i x + 2\left(\alpha_3 k_i^3 + \eta/k_i\right) t\right). \tag{2.9}$$

It therefore follows that the general structure of the 1-, 2- and 3-soliton solutions is respectively given by (after $\phi \to \mathrm{i}\phi$)

$$\begin{aligned}
\phi_{1\text{-sol}} &= \mathrm{i} ln\left(\frac{1 - a_1\rho_1}{1 + a_1\rho_1}\right), \\
\phi_{2\text{-sol}} &= \mathrm{i} ln\left(\frac{1 - a_1\rho_1 - a_2\rho_2 + a_1 a_2 a_{12}\rho_1\rho_2}{1 + a_1\rho_1 + a_2\rho_2 + a_1 a_2 a_{12}\rho_1\rho_2}\right), \\
\phi_{3\text{-sol}} &= \mathrm{i} ln\left(\frac{1 - \sum_{i=1}^3 a_i\rho_i + \sum_{i<j=1}^3 a_i a_j a_{ij}\rho_i\rho_j - a_1 a_2 a_3 a_{12} a_{13} a_{23}\rho_1\rho_2\rho_3}{1 + \sum_{i=1}^3 a_i\rho_i + \sum_{i<j=1}^3 a_i a_j a_{ij}\rho_i\rho_j + a_1 a_2 a_3 a_{12} a_{13} a_{23}\rho_1\rho_2\rho_3}\right)
\end{aligned} \tag{2.10}$$

where a_1, a_2 are constants and $a_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2$.

More general solutions (N-solitons and breathers) were found in [4–6] with same time dependence as in (2.9).

3. The mixed AKNS/Lund–Regge model

Let us consider another example involving $\mathcal{G} = \hat{sl}(2)$ and homogeneous gradation $Q = \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}$, $E^{(n)} = \lambda^n h$, $E = E^{(1)}$ and $A_0 = qE_\alpha + rE_{-\alpha}$ and the zero curvature representation of the form

$$\left[\partial_x + E + A_0, \partial_t + D_2^{(2)} + D_2^{(1)} + D_2^{(0)} + D_2^{(-1)}\right] = 0. \tag{3.1}$$

According to gradation Q, propose

$$D_2^{(j)} = \lambda^j(\alpha_j E_\alpha + \beta_j E_{-\alpha} + \sigma_j h), \qquad j = -1, 0, 1, 2 \tag{3.2}$$

In order to find solution for (3.1) we introduce variables $\tilde{\psi}$ and $\tilde{\chi}$ [8],

$$A_0 = qE_\alpha + rE_{-\alpha} = -\partial_x B B^{-1}, \qquad D_2^{(-1)} = \eta B E^{(-1)} B^{-1}, \qquad B = \mathrm{e}^{\tilde{\chi}E_{-\alpha}}\,\mathrm{e}^{\phi h}\,\mathrm{e}^{\tilde{\psi}E_\alpha} \tag{3.3}$$

which defines

$$q = -\partial_x\tilde{\psi}\,\mathrm{e}^{2\phi}, \qquad r = \tilde{\chi}^2\partial_x\tilde{\psi}\,\mathrm{e}^{2\phi} - \partial_x\tilde{\chi} \tag{3.4}$$

together with the subsidiary conditions for the non-local auxiliary field ϕ,

$$Tr(\partial_x B B^{-1} h) = \partial_x \phi - \tilde{\chi}\partial_x \tilde{\psi}\, e^{2\phi} = 0, \qquad Tr(B^{-1}\partial_t B h) = \partial_t \phi - \tilde{\psi}\partial_t \tilde{\chi}\, e^{2\phi} = 0. \quad (3.5)$$

Solution of constraints (3.5) leads to natural variables [9]

$$\psi = \tilde{\psi}\, e^{\phi}, \qquad \chi = \tilde{\chi}\, e^{\phi}. \quad (3.6)$$

Inserting (3.2) into (3.1) and collecting powers of λ, we find solution in terms of non-local fields ψ and χ

$$\begin{array}{llll} \sigma_2 = \text{const}, & \beta_2 = \alpha_2 = 0, & \sigma_1 = 0, & \sigma_0 = -1/2\sigma_2 r q \\ \beta_1 = \sigma_2 r, & \alpha_1 = \sigma_2 q, & \alpha_0 = -1/2\sigma_2 q_x, & \beta_0 = 1/2\sigma_2 r_x, \\ \alpha_{-1} = -2\eta\psi\, e^{\phi}, & \beta_{-1} = 2\eta(\chi + \psi\chi^2)\, e^{-\phi}, & \sigma_{-1} = \eta(1 + 2\psi\chi) & \end{array} \quad (3.7)$$

leading to the equations of motion

$$q_t + \tfrac{1}{2}\sigma_2(q_{xx} - 2q^2 r) - 2\alpha_{-1} = 0, \qquad r_t - \tfrac{1}{2}\sigma_2(r_{xx} - 2r^2 q) + 2\beta_{-1} = 0, \quad (3.8)$$

where q and r in variables ψ and χ reads

$$q = -\frac{\partial_x \psi}{1 + \psi\chi}\, e^{\phi} \qquad r = -\partial_x \chi\, e^{-\phi}. \quad (3.9)$$

Equations (3.8) represent a mixed system of AKNS (for $\eta = 0, \alpha_{-1} = \beta_{-1} = 0$) in variables q, r and the relativistic Lund–Regge (for $\sigma_2 = 0$) in variables ψ, χ.

$$\partial_t\left(\frac{\partial_x \psi}{\Delta}\right) + \psi\frac{\partial_t \chi \partial_x \psi}{\Delta^2} + 4\eta\psi = 0, \qquad \partial_x\left(\frac{\partial_t \chi}{\Delta}\right) + \chi\frac{\partial_t \chi \partial_x \psi}{\Delta^2} + 4\eta\chi = 0. \quad (3.10)$$

Again the terms proportional to α_{-1} and β_{-1} originate from the contribution of $D_2^{(-1)} = \eta B E^{(-1)} B^{-1}$ in (3.1) and the vacuum configuration is obtained for $\psi_{\text{vac}} = \chi_{\text{vac}} = q_{\text{vac}} = r_{\text{vac}} = 0$. The model is now characterized by $E^{(n)} = \lambda^n h$ and the vacuum solution of (3.1) yield

$$T_0 = \exp(t(\sigma_2 E^{(2)} + \eta E^{(-1)}))\exp(x E^{(1)}). \quad (3.11)$$

and therefore the spacetime dependence in ρ_i comes in the form

$$\rho_i = \exp\left(2k_i x + 2\left(\sigma_2 k_i^2 + \eta/k_i\right) t\right). \quad (3.12)$$

We have checked the solution for the composite model (3.8) to agree with the functional form of the one proposed in [9] with modified spacetime dependence given by (3.12), i.e.,

$$\psi = \frac{b\rho_2}{1 + \frac{k_1}{k_2}\Gamma\rho_1^{-1}\rho_2}, \qquad \chi = \frac{a\rho_1^{-1}}{1 + \frac{k_1}{k_2}\Gamma\rho_1^{-1}\rho_2}, \qquad e^{-\phi} = \frac{1 + \frac{k_1}{k_2}\Gamma\rho_1^{-1}\rho_2}{1 + \Gamma\rho_1^{-1}\rho_2} \quad (3.13)$$

where a and b are constants, $\Gamma = \frac{abk_2^2}{(k_1 - k_2)}$. In terms of AKNS field variables, from (3.9) we find

$$r = -\frac{2ak_1\rho_1^{-1}}{1 + \frac{abk_1k_2}{(k_1 - k_2)^2}\rho_1^{-1}\rho_2}, \qquad q = \frac{2bk_2\rho_2}{1 + \frac{abk_1k_2}{(k_1 - k_2)^2}\rho_1^{-1}\rho_2}. \quad (3.14)$$

4. The supersymmetric mKdV/sinh-Gordon model

Following the same line of reasoning, we now consider algebraic structures with half integer gradation [10]. Let $\hat{\mathcal{G}} = \hat{sl}(2,1)$, $Q = 2\lambda\frac{d}{d\lambda} + \frac{1}{2}h$ and $E^{(1)} = \lambda^{1/2}(h_1 + 2h_2) - (E_{\alpha_1} + \lambda E_{-\alpha_1})$. The graded structure can be decomposed as follows (see the appendix of [11]) for instance),

$$\begin{aligned}
\mathcal{K}_{\text{Bose}} &= \left\{K_1^{(2n+1)} = -\left(E_{\alpha_1}^{(n)} + E_{-\alpha_1}^{(n+1)}\right), K_2^{(2n+1)} = \mu_2 \cdot H^{(n+1/2)}\right\},\\
\mathcal{M}_{\text{Bose}} &= \left\{M_1^{(2n+1)} = -E_{\alpha_1}^{(n)} + E_{-\alpha_1}^{(n+1)}, M_2^{(2n)} = h_1^{(n)} = \alpha_1 \cdot H^{(n)}\right\},\\
\mathcal{K}_{\text{Fermi}} &= \left\{F_1^{(2n+3/2)} = \left(E_{\alpha_1+\alpha_2}^{(n+1/2)} - E_{\alpha_2}^{(n+1)}\right) + \left(E_{-\alpha_1-\alpha_2}^{(n+1)} - E_{-\alpha_2}^{(n+1/2)}\right),\right.\\
&\qquad \left. F_2^{(2n+1/2)} = -\left(E_{\alpha_1+\alpha_2}^{(n)} - E_{\alpha_2}^{(n+1/2)}\right) + \left(E_{-\alpha_1-\alpha_2}^{(n+1/2)} - E_{-\alpha_2}^{(n)}\right)\right\},\\
\mathcal{M}_{\text{Fermi}} &= \left\{G_1^{(2n+1/2)} = \left(E_{\alpha_1+\alpha_2}^{(n)} + E_{\alpha_2}^{(n+1/2)}\right) + \left(E_{-\alpha_1-\alpha_2}^{(n+1/2)} + E_{-\alpha_2}^{(n)}\right),\right.\\
&\qquad \left. G_2^{(2n+3/2)} = -\left(E_{\alpha_1+\alpha_2}^{(n+1/2)} + E_{\alpha_2}^{(n+1)}\right) + \left(E_{-\alpha_1-\alpha_2}^{(n+1)} + E_{-\alpha_2}^{(n+1/2)}\right)\right\},
\end{aligned} \tag{4.1}$$

where we have denoted $E_{\pm\alpha}^{(n)} = \lambda^n E_{\pm\alpha}$ and $H^{(n)} = \lambda^n H$ and $\alpha_i, \mu_i, i = 1, 2$ are respectively the simple roots and fundamental weights of $sl(2,1)$. In (4.1) we have denoted $\mathcal{K} = \mathcal{K}_{\text{Bose}} \cup \mathcal{K}_{\text{Fermi}}$ to be the Kernel of $E^{(1)}$, i.e., $[E^{(1)}, \mathcal{K}] = 0$ and $\mathcal{M}$ is its complement. The Lax operator is constructed as

$$L = \partial_x + E^{(1)} + A_{1/2} + A_0, \qquad A_0 = v M_2^{(0)}, \qquad A_{1/2} = \bar{\psi} G_1^{(1/2)}, \tag{4.2}$$

and the zero curvature representation reads

$$\left[\partial_x + E^{(1)} + A_{1/2} + A_0, \partial_t + D_3^{(3)} + D_3^{(5/2)} + \cdots + D_3^{(-1/2)} + D_3^{(-1)}\right] = 0. \tag{4.3}$$

In order to solve for the lowest grades $-1, -1/2$ of equation (4.3) we introduce the parametrization

$$D_3^{(-1)} = \eta B E^{(-1)} B^{-1}, \qquad A_0 = -\partial_x B B^{-1}, \qquad B = e^{\phi M_2^{(0)}} \tag{4.4}$$

together with the change of variables

$$D_3^{(-1/2)} = B j_{-1/2} B^{-1}, \qquad j_{-1/2} = \psi G_2^{(-1/2)}. \tag{4.5}$$

We propose the solution of the form

$$\begin{aligned}
D_3^{(3)} &= \alpha_3\left(h_1^{(3/2)} + 2h_2^{(3/2)} - E_{\alpha_1}^{(1)} - E_{-\alpha_1}^{(2)}\right),\\
D_3^{(0)} &= \alpha_1 M_2^{(0)},\\
D_3^{(1/2)} &= \beta_1 G_1^{(1/2)} + \beta_2 F_2^{(1/2)},\\
D_3^{(1)} &= \sigma_1 M_1^{(1)} + \sigma_2 K_1^{(1)} + \sigma_3 K_2^{(1)},\\
D_3^{(3/2)} &= \delta_1 G_2^{(3/2)} + \delta_2 F_1^{(3/2)},\\
D_3^{(2)} &= \mu_1 M_2^{(2)},\\
D_3^{(5/2)} &= \nu_1 G_1^{(5/2)} + \nu_2 F_2^{(5/2)},\\
D_3^{(-1/2)} &= \beta_{-1} G_1^{(-1/2)} + \beta_{-2} F_1^{(-1/2)},\\
D_3^{(-1)} &= \sigma_{-1} M_1^{(-1)} + \sigma_{-2} K_1^{(-1)} + \sigma_{-3} K_2^{(-1)}.
\end{aligned} \tag{4.6}$$

where the coefficients are given by

$$\begin{aligned}
&\alpha_1 = \tfrac{1}{4}\partial_x^2 v + \tfrac{3}{4} v\bar{\psi}\partial_x\bar{\psi} - \tfrac{1}{2}v^3, \qquad \beta_1 = \tfrac{1}{4}\partial_x^2\bar{\psi} - \tfrac{1}{2}v^2\bar{\psi}, \qquad \beta_2 = \tfrac{1}{4}(v\partial_x\bar{\psi} - \bar{\psi}\partial_x v),\\
&\sigma_1 = \tfrac{1}{2}\partial_x v, \qquad \sigma_2 = \tfrac{1}{2}(\bar{\psi}\partial_x\bar{\psi} - v^2), \qquad \sigma_3 = -\tfrac{1}{2}\bar{\psi}\partial_x\psi \qquad \delta_1 = -\tfrac{1}{2}\partial_x\bar{\psi},\\
&\delta_2 = -\tfrac{1}{2}v\bar{\psi}, \qquad \mu_1 = v, \qquad \nu_1 = \bar{\psi}, \qquad \nu_2 = 0, \qquad \beta_{-1} = \psi\cosh\phi,\\
&\beta_{-2} = -\psi\sinh\phi, \qquad \sigma_{-1} = \eta\sinh 2\phi, \qquad \sigma_{-2} = \eta\cosh 2\phi, \qquad \sigma_{-3} = \eta,
\end{aligned} \tag{4.7}$$

where α_3 and η are arbitrary constants. The equations of motion are given by grades 0, $\pm 1/2$ projections of (4.3), i.e.,

$$\begin{aligned}
&\partial_t\partial_x\phi = \frac{\alpha_3}{4}\left[\partial_x^4\phi - 6(\partial_x\phi)^2\partial_x^2\phi + 3\bar{\psi}\partial_x(\partial_x\phi\partial_x\bar{\psi})\right] + 2\eta[\sinh(2\phi) + \bar{\psi}\psi\sinh(\phi)],\\
&\partial_{t_3}\bar{\psi} = \frac{\alpha_3}{4}\left[\partial_x^3\bar{\psi} - 3\partial_x\phi\partial_x(\partial_x\phi\bar{\psi})\right] + 2\eta\psi\cosh(\phi),\\
&\partial_x\psi = 2\bar{\psi}\cosh(\phi).
\end{aligned} \tag{4.8}$$

Observe that for $\eta = 0$ equations (4.8) corresponds to the $N = 1$ super mKdV equation if we identify $v = -\partial_x\phi$ and for $\alpha_3 = 0$ they correspond to the $N = 1$ super sinh-Gordon.

The soliton solutions are parametrized in terms of tau functions as

$$\phi = ln\left(\frac{\tau_1}{\tau_0}\right), \qquad \bar{\psi} = \frac{\tau_3}{\tau_1} + \frac{\tau_2}{\tau_0}. \tag{4.9}$$

The one-soliton solution for the $N = 1$ super sinh-Gordon and mKdV equations is given by

$$\begin{aligned}
&\tau_0 = 1 - \tfrac{1}{2}b_1\rho_1, \qquad \tau_1 = 1 + \tfrac{1}{2}b_1\rho_1,\\
&\tau_2 = c_1k_2\rho_2^{-1} + b_1c_1\sigma_{1,2}\rho_1\rho_2^{-1}, \qquad \tau_3 = c_1k_2\rho_2^{-1} - b_1c_1\sigma_{1,2}\rho_1\rho_2^{-1},
\end{aligned} \tag{4.10}$$

where $\sigma_{1,2} = \frac{1}{2}k_2\frac{(k_1+k_2)}{(k_1-k_2)}$, b_1, c_1 are bosonic and Grassmaniann constants respectively and ρ_i carries the spacetime dependence for the sinh-Gordon and mKdV, respectively,

$$\rho_i^{\text{mKdV}} = \exp\left(2k_ix + 2\left(\alpha_3k_i^3\right)t\right), \qquad \rho_i^{s-G} = \exp\left(2k_ix + 2\left(\frac{\eta}{k_i}\right)t\right). \tag{4.11}$$

Note however that the introduction of the $D_{-1}^{(-1/2)}$ and $D_{-1}^{(-1)}$ terms changes the vacuum configuration such that

$$T_0 = \exp(xE^{(1)})\exp(\alpha_3E^{(3)} + \eta E^{(-1)})t) \tag{4.12}$$

which induces modification in the spacetime dependence of equations (4.8) as

$$\rho_i = \exp(2k_ix)\exp\left(2\left(\alpha_3k_i^3 + \frac{\eta}{k_i}\right)t\right). \tag{4.13}$$

In fact we have verified explicitly that (4.10) with (4.13) satisfies the equations of motion (4.8). The same was verified for the two soliton solution

$$\begin{aligned}
\tau_0 &= 1 - \tfrac{1}{2}b_1\rho_1 - \tfrac{1}{2}b_2\rho_2 + b_1b_2\rho_1\rho_2\alpha_{1,2}\\
&\quad + c_1c_2\rho_3^{-1}\rho_4^{-1}(\beta_{3,4} - b_1\rho_1\delta_{1,3,4} - b_2\rho_2\delta_{2,3,4} + b_1b_2\rho_1\rho_2\theta_{1,2,3,4}),\\
\tau_1 &= 1 + \tfrac{1}{2}b_1\rho_1 + \tfrac{1}{2}b_2\rho_2 + b_1b_2\rho_1\rho_2\alpha_{1,2}\\
&\quad + c_1c_2\rho_3^{-1}\rho_4^{-1}(\beta_{3,4} + b_1\rho_1\delta_{1,3,4} + b_2\rho_2\delta_{2,3,4} + b_1b_2\rho_1\rho_2\theta_{1,2,3,4}),\\
\tau_2 &= c_1\rho_3^{-1}(k_3 + b_1\rho_1\sigma_{1,3} + b_2\rho_2\sigma_{2,3} + b_1b_2\rho_1\rho_2\lambda_{1,2,3})\\
&\quad + c_2\rho_4^{-1}(k_4 + b_1\rho_1\sigma_{1,4} + b_2\rho_2\sigma_{2,4} + b_1b_2\rho_1\rho_2\lambda_{1,2,4}),\\
\tau_3 &= c_1\rho_3^{-1}(k_3 - b_1\rho_1\sigma_{1,3} - b_2\rho_2\sigma_{2,3} + b_1b_2\rho_1\rho_2\lambda_{1,2,3})\\
&\quad + c_2\rho_4^{-1}(k_4 - b_1\rho_1\sigma_{1,4} - b_2\rho_2\sigma_{2,4} + b_1b_2\rho_1\rho_2\lambda_{1,2,4}),
\end{aligned} \tag{4.14}$$

where

$$\alpha_{1,2} = \frac{1}{4}\frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \qquad \beta_{3,4} = k_3k_4\frac{(k_3-k_4)}{(k_3+k_4)^2},$$
$$\delta_{j,3,4} = \frac{k_3k_4}{2}\frac{(k_3-k_4)}{(k_3+k_4)^2}\frac{(k_j+k_3)}{(k_j-k_3)}\frac{(k_j+k_4)}{(k_j-k_4)} \qquad (j=1,2),$$
$$\sigma_{j,k} = \frac{k_k}{2}\frac{(k_j+k_k)}{(k_j-k_k)} \qquad (j=1,2) \qquad (k=3,4), \tag{4.15}$$
$$\lambda_{1,2,j} = \frac{k_j}{4}\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{(k_1+k_j)}{(k_1-k_j)}\frac{(k_2+k_j)}{(k_2-k_j)}, \qquad (j=3,4),$$
$$\theta_{1,2,3,4} = \frac{k_3k_4}{4}\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{(k_1+k_3)}{(k_1-k_3)}\frac{(k_2+k_3)}{(k_2-k_3)}\frac{(k_3-k_4)}{(k_3+k_4)^2}\frac{(k_1+k_4)}{(k_1-k_4)}\frac{(k_2+k_4)}{(k_2-k_4)},$$

b_1, b_2 are bosonic constants and c_1, c_2 are Grassmaniann constants with ρ_i given by (4.13).

5. The supersymmetric Lund–Regge/AKNS model

In this section we consider the Lie superalgebra $\hat{\mathcal{G}} = \hat{sl}(2,1)$ with homogeneous gradation, $Q = \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}$ and (see for instance [12])

$$E^{(n)} = (\alpha_1+\alpha_2)\cdot H^{(n)}, \qquad \alpha_1, \alpha_2 \quad \text{are simple roots of sl(2,1).} \tag{5.1}$$

The Lax operator is then

$$L = \partial_x + E^{(1)} + A_0, \qquad A_0 = b_1E_{\alpha_1} + \bar{b}_1E_{-\alpha_1} + F_1E_{\alpha_2} + \bar{F}_1E_{-\alpha_2}. \tag{5.2}$$

We search for the solution of

$$\left[\partial_x + E^{(1)} + A_0, \partial_t + D_2^{(2)} + D_2^{(1)} + D_2^{(0)} + D_2^{(-1)}\right] = 0. \tag{5.3}$$

Decomposing (5.3) grade by grade, we find

$$\begin{aligned} D_2^{(2)} &= a_2\lambda^2\alpha_1\cdot H,\\ D_2^{(1)} &= g_1\lambda E_{\alpha_1} + m_1\lambda E_{-\alpha_1} + n_1\lambda E_{-\alpha_2} + o_1\lambda E_{\alpha_2}\\ D_{2\mathcal{M}}^{(0)} &= g_0E_{\alpha_1} + m_0E_{-\alpha_1} + n_0E_{-\alpha_2} + o_0E_{\alpha_2},\\ D_{2\mathcal{K}}^{(0)} &= a_0\alpha_1\cdot H + c_0\alpha_2\cdot H + d_0E_{\alpha_1+\alpha_2} + e_0E_{-\alpha_1-\alpha_2}.\end{aligned} \tag{5.4}$$

where $D_2^{(0)} = D_{2\mathcal{M}}^{(0)} + D_{2\mathcal{K}}^{(0)}$ and

$$\begin{aligned} &g_1 = a_2b_1 && m_1 = a_2\bar{b}_1, && o_1 = a_2F_1, && n_1 = a_2\bar{F}_1,\\ &g_0 = a_2\partial_xb_1, && m_0 = -a_2\partial_x\bar{b}_1, && n_0 = a_2\partial_x\bar{F}_1, && o_0 = -a_2\partial_xF_1,\\ &d_0 = -a_2F_1b_1, && e_0 = -a_2\bar{F}_1\bar{b}_1, && a_0 = -a_2b_1\bar{b}_1, && c_0 = -a_2(b_1\bar{b}_1 + F_1\bar{F}_1).\end{aligned}$$

In order to solve the grade -1 projection of equation (5.3) we introduce the $sl(2,1)$ variables [12] as

$$A_0 = -\partial_xBB^{-1} = b_1E_{\alpha_1} + \bar{b}_1E_{-\alpha_1} + F_1E_{\alpha_2} + \bar{F}_1E_{-\alpha_2}, \tag{5.5}$$

where

$$B = \mathrm{e}^{\tilde{\chi}E_{-\alpha_1}}\,\mathrm{e}^{\tilde{f}_1E_{-\alpha_1-\alpha_2}}\,\mathrm{e}^{\tilde{f}_2E_{\alpha_2}}\,\mathrm{e}^{\varphi_1(\alpha_1+\alpha_2)\cdot H-\varphi_2\alpha_2\cdot H}\,\mathrm{e}^{\tilde{g}_2E_{-\alpha_2}}\,\mathrm{e}^{\tilde{g}_1E_{\alpha_1+\alpha_2}}\,\mathrm{e}^{\tilde{\psi}E_{\alpha_1}} \tag{5.6}$$

and

$$\begin{aligned} D_{2\mathcal{M}}^{(-1)} = \eta BE^{(-1)}B^{-1} &= -\eta\psi\,\mathrm{e}^{\frac{1}{2}(\phi_1+\phi_2)}\lambda^{-1}E_{\alpha_1} + \eta f_2(1+\psi\chi)\,\mathrm{e}^{-\frac{1}{2}\phi_1}\lambda^{-1}E_{\alpha_2},\\ &+\eta(\chi + f_1f_2 + \psi\chi^2 + \psi\chi f_1f_2)\,\mathrm{e}^{-\frac{1}{2}(\phi_1+\phi_2)}\lambda^{-1}E_{-\alpha_1}\\ &-\eta(g_2+\psi f_1)\,\mathrm{e}^{\frac{1}{2}\phi_1}\lambda^{-1}E_{-\alpha_2}\end{aligned} \tag{5.7}$$

written in the natural variables

$$\tilde{\psi} = \psi\, e^{-\frac{\varphi_1+\varphi_2}{2}}, \qquad \tilde{g_1} = g_1\, e^{-\frac{\varphi_2}{2}}, \qquad \tilde{f_1} = f_1\, e^{-\frac{\varphi_2}{2}}$$
$$\tilde{\chi} = \chi\, e^{-\frac{\varphi_1+\varphi_2}{2}}, \qquad \tilde{g_2} = g_2\, e^{-\frac{\varphi_1}{2}}, \qquad \tilde{f_2} = f_2\, e^{-\frac{\varphi_1}{2}}. \tag{5.8}$$

Here, $\psi, \chi, \varphi_i, i = 1, 2$ and $f_i, g_i, i = 1, 2$ are bosonic and fermionic fields, respectively. The absence of Cartan subalgebra h_1, h_2 and $E_{\pm(\alpha_1+\alpha_2)}$ (i.e. in $\mathcal{K}$) in the rhs of (5.5) leads to the following subsidiary constraints:

$$\begin{aligned}
\partial_t f_1 &= \frac{1}{2} f_1 \partial_t \varphi_2 + g_2 \left[\partial_t \chi - \frac{1}{2}\chi(\partial_t\varphi_1 + \partial_t\varphi_2)\right], \\
\partial_t g_1 &= \psi\partial_t f_2 + \frac{1}{2} g_1 \partial_t\varphi_2 - \frac{1}{2}\psi f_2 \partial_t\varphi_1, \\
\partial_x f_1 &= \chi\partial_x g_2 + \frac{1}{2} f_1\partial_x\varphi_2 - \frac{1}{2}\chi g_2\partial_x\varphi_1, \\
\partial_x g_1 &= \frac{1}{2} g_1\partial_x\varphi_2 + f_2\left[\partial_x\psi - \frac{1}{2}\psi(\partial_x\varphi_1 + \partial_x\varphi_2)\right], \\
\partial_t\varphi_1 &= \frac{\psi\left[\partial_t\chi(1 + g_2 f_2) + \frac{1}{2}\chi g_2\partial_t f_2\right]}{1 + \psi\chi\left(1 + \frac{5}{4} g_2 f_2\right)}, \\
\partial_t\varphi_2 &= \frac{\psi\partial_t\chi\left(1 + \frac{3}{2} g_2 f_2\right) - g_2\partial_t f_2 - \frac{1}{2}\psi\chi g_2\partial_t f_2}{1 + \psi\chi\left(1 + \frac{5}{4} g_2 f_2\right)}, \\
\partial_x\varphi_1 &= \frac{\chi\left[\partial_x\psi(1 + g_2 f_2) + \frac{1}{2}\psi\partial_x g_2 f_2\right]}{1 + \psi\chi\left(1 + \frac{5}{4} g_2 f_2\right)}, \\
\partial_x\varphi_2 &= \frac{\chi\partial_x\psi\left(1 + \frac{3}{2} g_2 f_2\right) + \left(\frac{1}{2}\psi\chi + 1\right) f_2\partial_x g_2}{1 + \psi\chi\left(1 + \frac{5}{4} g_2 f_2\right)}.
\end{aligned} \tag{5.9}$$

Moreover equation (5.5) yields

$$\begin{aligned}
\bar{b}_1 &= \bar{J}_{-\alpha_1} = -\frac{e^{\frac{1}{2}(\varphi_1+\varphi_2)}}{1 + f_2 g_2}\left(\partial_x\psi - \frac{1}{2}\psi(\partial_x\varphi_1 + \partial_x\varphi_2)\right), \\
F_1 &= \bar{J}_{-\alpha_2} = -e^{-\frac{1}{2}\varphi_1}\left(\partial_x f_2 + \frac{1}{2} f_2\partial_x\varphi_1\right), \\
b_1 &= -e^{-\frac{1}{2}(\varphi_1+\varphi_2)}\Big(\partial_x\chi + \frac{1}{2}\chi(\partial_x\varphi_1 + \partial_x\varphi_2) - \chi f_2\partial_x g_2 - \frac{1}{2}\chi\partial_x\varphi_1 g_2 f_2 \\
&\qquad - e^{\frac{1}{2}\varphi_1} f_1 \bar{J}_{-\alpha_2} + \chi^2 e^{-\frac{1}{2}(\varphi_1-\varphi_2)}\bar{J}_{-\alpha_1}\Big), \\
\bar{F}_1 &= -e^{\frac{1}{2}\varphi_1}\left(\partial_x g_2 - \frac{1}{2} g_2\partial_x\varphi_1 + e^{-\frac{1}{2}(\varphi_1-\varphi_2)} f_1 \bar{J}_{-\alpha_1}\right).
\end{aligned} \tag{5.10}$$

Solving the zero grade component of (5.3), we find the equations of motion,

$$\begin{aligned}
&\partial_{t_2} b_1 + a_2(\partial_x^2 b_1 - 2\big(b_1\bar{b}_1 + F_1\bar{F}_1\big)b_1) + m_{-1} = 0, \\
&\partial_{t_2}\bar{b}_1 - a_2\left(\partial_x^2\bar{b}_1 - 2(b_1\bar{b}_1 + F_1\bar{F}_1)\bar{b}_1\right) - g_{-1} = 0, \\
&\partial_{t_2} F_1 - a_2\left(\partial_x^2 F_1 - 2b_1\bar{b}_1 F_1\right) - n_{-1} = 0, \\
&\partial_{t_2}\bar{F}_1 + a_2\left(\partial_x^2\bar{F}_1 - 2b_1\bar{b}_1\bar{F}_1\right) + o_{-3} = 0,
\end{aligned} \tag{5.11}$$

where

$$g_{-1} = -\eta\psi\, e^{\frac{1}{2}(\phi_1+\phi_2)}, \tag{5.12}$$

$$m_{-1} = \eta(\chi + f_1 f_2 + \psi\chi f_1 f_2 + \chi f_2 g_2 + \psi\chi^2)\,\mathrm{e}^{-\frac{1}{2}(\phi_1+\phi_2)},$$
$$n_{-1} = -\eta(g_2 + \psi f_1)\,\mathrm{e}^{\frac{1}{2}\phi_1}, \tag{5.13}$$
$$o_{-1} = \eta f_2(1 + \psi\chi)\,\mathrm{e}^{-\frac{1}{2}\phi_1}.$$

Following the same argument as in the pure bosonic case, the vacuum configuration is obtained from

$$T_0 = \exp(xE^{(1)})\exp((\alpha_2 E^{(2)} + \eta E^{(-1)})t) \tag{5.14}$$

which leads to spacetime dependence

$$\rho_i = \exp(k_i x)\exp\left(-\left(\alpha_2 k_i^2 + \frac{\eta}{k_i}\right)t\right). \tag{5.15}$$

Following the soliton solutions for the Lund–Regge model obtained in [12] we have verified solutions for equations (5.11) to be

$$b_1 = \frac{k_1\rho_1^{-1}}{\tau_0}, \qquad \bar{b}_1 = -\frac{k_2\rho_2}{\tau_0}, \qquad F_1 = -a_2\frac{k_2\rho_2}{\tau_0}, \qquad \bar{F}_1 = a_1\frac{k_1\rho_1^{-1}}{\tau_0},$$
$$\psi = \frac{\rho_1}{\tau_0}\left(1 - \frac{bk_1\rho_1^{-1}\rho_2}{2(k_1-k_2)\left(1+\frac{k_1}{k_2}\rho_1^{-1}\rho_2\right)}\right), \qquad \chi = \frac{\rho_2}{\tau_0}\left(1 - \frac{bk_2\rho_1^{-1}\rho_2}{2(k_1-k_2)\left(1+\frac{k_1}{k_2}\rho_1^{-1}\rho_2\right)}\right),$$
$$g_1 = a_2\frac{k_1\rho_1^{-1}\rho_2}{(k_1-k_2)\tau_0}\,\mathrm{e}^{-\frac{1}{2}\phi_1}, \qquad f_1 = a_1\frac{k_1\rho_1^{-1}\rho_2}{(k_1-k_2)\tau_0}\,\mathrm{e}^{-\frac{1}{2}\phi_1}, \qquad g_2 = a_1\frac{\rho_1^{-1}}{\tau_0}\,\mathrm{e}^{-\frac{1}{2}\phi_2},$$
$$f_2 = a_2\frac{\rho_2}{\tau_0}\,\mathrm{e}^{-\frac{1}{2}\phi_2}, \qquad \mathrm{e}^{\frac{1}{2}(\phi_1+\phi_2)} = \frac{1+a_3\rho_1\rho_2}{\tau_0}, \qquad \mathrm{e}^{\frac{1}{2}(\phi_1-\phi_2)} = \frac{1+\bar{a}_3\rho_1^{-1}\rho_2}{\tau_0}, \tag{5.16}$$

where a_1, a_2 and b are Grassmaniann and bosonic constants respectively, ρ_i, $i = 1, 2$ are given by (5.15) and

$$a_3 = \frac{k_1}{k_2}\Gamma_0\left(1 - b\frac{(k_1+k_2)}{2k_1}\right), \qquad \bar{a}_3 = \Gamma_0\left(1 + b\frac{(k_1-3k_2)}{2k_2}\right),$$
$$\Gamma = (1 - a_1 a_2)\Gamma_0, \qquad \Gamma_0 = \frac{k_1 k_2}{(k_1-k_2)^2}, \qquad \tau_0 = 1 + \Gamma\rho_1^{-1}\rho_2. \tag{5.17}$$

6. General case

We now consider a mixed hierarchy associated with a general affine Lie algebra $\hat{\mathcal{G}} = \oplus\mathcal{G}_i, [Q, \mathcal{G}_i] = i\mathcal{G}_i$ and constant grade one semi-simple element E such that $\hat{\mathcal{G}} = \mathcal{M}\oplus\mathcal{K}, [E, \mathcal{K}] = 0$ with the symmetric space structure,

$$[\mathcal{K},\mathcal{K}] \subset \mathcal{K}, \qquad [\mathcal{K},\mathcal{M}] \subset \mathcal{M}, \qquad [\mathcal{M},\mathcal{M}] \subset \mathcal{K} \tag{6.1}$$

with equations of motion involving time evolution with two indices, $t_{n,m}$ defined from the zero curvature representation

$$\left[\partial_x + E + A_0, \partial_{t_{n,m}} + D^{(n)} + D^{(n-1)} + \cdots + D^{(0)} + D^{(-1)} + \cdots + D^{(-m+1)} + D^{(-m)}\right] = 0. \tag{6.2}$$

Equation (6.2) leads to

$$[E, D^{(n)}] = 0, \tag{6.3}$$

$$[E, D^{(n-1)}] + [A_0, D^{(n)}] + \partial_x D^{(n)} = 0, \tag{6.4}$$

$$\vdots$$

$$[E, D^{(n-i)}] + [A_0, D^{(n-i+1)}] + \partial_x D^{(n-i+1)} = 0, \tag{6.5}$$

$$\vdots$$

$$[E, D^{(-1)}] + [A_0, D^{(0)}] + \partial_x D^{(0)} - \partial_{t_{n,m}} A_0 = 0, \tag{6.6}$$

$$[E, D^{(-2)}] + [A_0, D^{(-1)}] + \partial_x D^{(-1)} = 0, \tag{6.7}$$

$$\vdots$$

$$[E, D^{(-j-1)}] + [A_0, D^{(-j)}] + \partial_x D^{(-j)} = 0, \tag{6.8}$$

$$\vdots$$

$$[A_0, D^{(-m)}] + \partial_x D^{(-m)} = 0. \tag{6.9}$$

In order to solve equations (6.3)–(6.9) we have to start from both ends, i.e. from (6.3) towards (6.6), using the symmetric space structure (6.1), we project each equation into $\mathcal{K}$ and $\mathcal{M}$ subspaces to obtain $D^{(i)}_{\mathcal{K}}, i = 1, \ldots, n$ and $D^{(i)}_{\mathcal{M}}, i = 0, \ldots, n$. On the other hand, starting from (6.9) upwards, we find a solution for $D^{(-j)}_{\mathcal{K}}$ and $D^{(-j)}_{\mathcal{M}}$, $j = 1, \ldots, m$ which is non-local in the fields in A_0. For the particular case when $m = 1$, we have seen that there is a set of variables within a group element B that solves (6.9) locally for $m = 1$.

Inserting $D^{(-1)}$ in (6.6) and projecting in $\mathcal{K}$ we find $D^{(0)}_{\mathcal{K}}$ which in turn determines the time evolution as the projection of (6.6) in $\mathcal{M}$. Following the same arguments given before, the spacetime dependence of such generalized mixed model is expected to be of the form

$$\rho_i = \exp(k_i x) \exp\left(\left(\alpha_i k_i^n + \eta k_i^{-m}\right) t\right). \tag{6.10}$$

As a conclusion, we have proposed a zero curvature representation for mixed integrable models associated with $\hat{sl}(2)$ and $\hat{sl}(2,1)$ affine Lie algebras. We have also shown that their soliton solutions follow from the dressing method and with spacetime dependence specified from its vacuum structure. Other more complicated examples deserve to be investigated following the same line of thought.

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References

[1] Chodos A 1980 *Phys. Rev.* D **21** 2818

[2] Aratyn H, Gomes J F, Nissimov E, Pacheva S and Zimerman A H 2000 Symmetry flows, conservation laws and dressing approach to the integrable models *Proc. NATO Advanced Research Workshop on Integrable Hierarchies and Modern Physical Theories (NATO ARW-UIC 2000), Chicago (2000)* (arXiv:nlin/0012042)

[3] Aratyn H, Gomes J F and Zimerman A H 2003 *J. Geom. Phys.* **46** 21 (arXiv:hep-th/0107056)

[4] Konno K, Kameyama W and Sanuki H 1974 *J. Phys. Soc. Japan* **37** 171

[5] Chen D-Y, Zhang D-J and Deng S-F 2002 *J. Phys. Soc. Japan* **71** 658

[6] Leblond H, Melnikov I V and Mihalache D 2008 *Phys. Rev.* A **78** 043802

[7] Ferreira L A, Miramontes J L and Sanchez-Guillen J 1997 *J. Math. Phys.* **38** 882 (arXiv:hep-th/9606066)

[8] Aratyn H, Ferreira L A, Gomes J F and Zimerman A H 2000 *J. Phys. A: Math. Gen.* **33** L331 (arXiv:nlin/0007002)

[9] Cabrera-Carnero I, Gomes J F, Gueuvoghlanian E P, Sotkov G M and Zimerman A H 2001 *Proc. 7th Int. Wigner Symp. (Wigsym 7), College Park, MA, 2001* (arXiv:hep-th/0109117)
[10] Aratyn H, Gomes J F and Zimerman A H 2004 *Nucl. Phys.* B **676** 537 (arXiv:hep-th/0309099)
[11] Gomes J F, Ymai L H and Zimerman A H 2006 *Phys. Lett.* A **359** 630 (arXiv:hep-th/0607107)
[12] Aratyn H, Gomes J F, de Castro G M, Silka M B and Zimerman A H 2005 *J. Phys. A: Math. Gen.* **38** 9341 (arXiv:hep-th/0508008)